

Uniqueness of the ground state in the Feshbach renormalization analysis

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Abstract

In the operator theoretic renormalization analysis introduced by Bach, Fröhlich, and Sigal we prove uniqueness of the ground state.

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1 Introduction

In 1998 Bach, Fröhlich, and Sigal (BFS) introduced an operator theoretic renormalization group scheme [2] to analyze certain field theoretic models. This scheme was applied in [2] and refined by Bach, Chen, Fröhlich, and Sigal [1] (see also the work of Griesemer and Hasler [7]). More recently the renormalization group analysis was applied in [8, 14, 4, 10, 11, 12, 5]. If there is a positivity improving representation of the semigroup generated by the Hamiltonian then a ground state can often be shown to be unique for arbitrary coupling constant (if it exists). There are other methods (for example [3], [15]) that have been used to show that in some models with regular infrared behavior or minimal coupling, the multiplicity of the ground state is the same as for the electronic Hamiltonian for small coupling. We include the usual model of non-relativistic quantum electrodynamics (QED) as a model with regular infrared behavior (at least for its behavior near the ground state

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energy). This occurs because of an approximate gauge invariance. In the present paper we show that for small coupling, and under rather weak hypotheses, the renormalization group analysis leads to uniqueness of the ground state if the electronic Hamiltonian has a unique ground state even in certain QED models without any gauge invariance.

In Section 2 we state the main result in Theorem 2.1. In Section 3 we apply Theorem 2.1 to explicit models of non-relativistic QED. In the Appendix we apply Theorem 2.1 to operator theoretic renormalization as defined in [1].

2 An Abstract Result

We give the necessary hypotheses for an abstract result and then specialize to get to the applications in the next section. A further explicit application based directly on [1] is in an Appendix. Thus suppose \mathcal{H} is a separable Hilbert space and T_0 a self-adjoint operator in \mathcal{H} with spectrum $\sigma(T_0) = [0, \infty)$ such that T_0 has an eigenvalue at 0 of multiplicity m (thus $m > 0$). Suppose $\{\Gamma_t\}_{t>0}$ is a group of unitary scale transformations on \mathcal{H} satisfying $\Gamma_s \Gamma_u = \Gamma_{su}$ and that T_0 is such that $\Gamma_s T_0 \Gamma_s^{-1} = sT_0$. We define $\mathcal{H}_{\text{red}} := 1_{[0,1]}(T_0)\mathcal{H}$.

We assume we have a sequence of bounded operators $\{H_n\}_{n=1}^\infty$ on \mathcal{H}_{red} , such that the members of the sequence $\{H_n\}_{n=1}^\infty$ are related by the fact that H_{n+1} is up to a constant multiple and a scale transformation a Feshbach transform of H_n , that is we assume that (2.3) below holds. To state this more precisely we need some further notation. Suppose χ and $\bar{\chi}$ are non-negative Borel functions on \mathbb{R} satisfying $\chi^2 + \bar{\chi}^2 = 1$ with $\chi(x) = 1$ for $x \in [0, a]$ for some $a \in (0, 1]$ and $\chi(x) = 0$ for $x \geq 1$. We assume that χ is non-increasing. We write $H_n = T_n + W_n$ where both T_n and W_n are bounded operators on \mathcal{H}_{red} . We assume T_n commutes with T_0 . For any $t > 0$ let $\chi_t(x) = \chi(t^{-1}x)$ and similarly for $\bar{\chi}_t$. Choose $0 < \rho < a$. Define

$$\bar{H}_n := (T_n + \bar{\chi}_\rho(T_0)W_n\bar{\chi}_\rho(T_0))|_{\text{Ran}\bar{\chi}_\rho(T_0) \cap \mathcal{H}_{\text{red}}}. \quad (2.1)$$

We assume that \bar{H}_n and T_n are invertible on $\text{Ran}\bar{\chi}_\rho(T_0) \cap \mathcal{H}_{\text{red}}$ with bounded inverses and define

$$F_n := T_n + \chi_\rho W_n \chi_\rho - \chi_\rho W_n \bar{\chi}_\rho (\bar{H}_n)^{-1} \bar{\chi}_\rho W_n \chi_\rho \quad (2.2)$$

where here the arguments of χ_ρ and $\bar{\chi}_\rho$ are both T_0 . Then we assume

$$H_{n+1} = \rho^{-1} \Gamma_\rho F_n \Gamma_\rho^{-1}. \quad (2.3)$$

This equality is on the subspace \mathcal{H}_{red} .

Note that $\Gamma_\rho^{-1} : \mathcal{H}_{\text{red}} \rightarrow 1_{[0,\rho]}(T_0)\mathcal{H}$ and that F_n preserves the latter space. Finally $\Gamma_\rho : 1_{[0,\rho]}(T_0)\mathcal{H} \rightarrow \mathcal{H}_{\text{red}}$. A crucial component of our analysis is the fact shown in [7] that in our context $\chi_\rho(T_0)$ is an isomorphism of $\ker H_n$ onto $\ker F_n$ and thus $\Gamma_\rho \chi_\rho(T_0)$ is an isomorphism of $\ker H_n$ onto $\ker H_{n+1}$. (It was shown in [1] that $\chi_\rho(T_0)$ is injective as a

map from $\ker H_n \rightarrow \ker F_n$ and this is all we will use below.) It follows that $(\Gamma_\rho \chi_\rho(T_0))^n$ is an isomorphism of $\ker H_1$ onto $\ker H_{n+1}$.

Theorem 2.1. *Given the hypotheses and definitions above, we make the following additional assumptions. There exist $\delta_0 > 0$ and $\{a_n\}_{n=1}^\infty \in \ell^2(\mathbb{N})$, such that*

$$(a) \quad \|T_n \psi\| \geq \delta_0 \|T_0 \psi\| - a_n \|\psi\| \text{ for all } \psi \in \mathcal{H}_{\text{red}},$$

$$(b) \quad \|W_n\| \leq a_n.$$

Then $\ker H_1$ is at most m dimensional.

Proof. Suppose $\ker H_1$ is at least $m + 1$ dimensional. Then there exists a non-zero vector $\psi_1 \in \ker H_1$ so that $1_{\{0\}}(T_0)\psi_1 = 0$. It follows that $(\Gamma_\rho \chi_\rho(T_0))^n \psi_1$ is a non-zero vector in $\ker H_{n+1}$. Note that because $\rho < a$, $\chi_\rho \chi = \chi_\rho$. This gives $(\Gamma_\rho \chi_\rho(T_0))^n = \Gamma_{\rho^n} \chi_{\rho^n}(T_0)$. Now it is readily verified that

$$1_{\{0\}}(T_0) + \sum_{j=n}^{\infty} (1_{[a\rho^{j+1}, \infty)}(T_0) \chi_{\rho^j}(T_0))^2 \geq \chi_{\rho^n}(T_0)^2. \quad (2.4)$$

Thus

$$\begin{aligned} (a\rho)^2 \|\Gamma_{\rho^n} \chi_{\rho^n}(T_0) \psi_1\|^2 &= (a\rho)^2 \|\chi_{\rho^n}(T_0) \psi_1\|^2 \\ &\leq (a\rho)^2 \sum_{j=n}^{\infty} \|1_{[a\rho^{j+1}, \infty)}(T_0) \chi_{\rho^j}(T_0) \psi_1\|^2 \\ &\leq \sum_{j=n}^{\infty} \rho^{-2j} \|T_0 1_{[a\rho^{j+1}, \infty)}(T_0) \chi_{\rho^j}(T_0) \psi_1\|^2 \\ &\leq \sum_{j=n}^{\infty} \rho^{-2j} \|T_0 \chi_{\rho^j}(T_0) \psi_1\|^2 \\ &= \sum_{j=n}^{\infty} \rho^{-2j} \|\Gamma_{\rho^j} T_0 \chi_{\rho^j}(T_0) \psi_1\|^2 \\ &= \sum_{j=n}^{\infty} \|T_0 \Gamma_{\rho^j} \chi_{\rho^j}(T_0) \psi_1\|^2. \end{aligned} \quad (2.5)$$

By hypothesis

$$0 = \|H_{j+1} \Gamma_{\rho^j} \chi_{\rho^j}(T_0) \psi_1\| \geq \delta_0 \|T_0 \Gamma_{\rho^j} \chi_{\rho^j}(T_0) \psi_1\| - 2a_{j+1} \|\Gamma_{\rho^j} \chi_{\rho^j}(T_0) \psi_1\|.$$

Thus

$$\|T_0 \Gamma_{\rho^j} \chi_{\rho^j}(T_0) \psi_1\| \leq 2\delta_0^{-1} a_{j+1} \|\Gamma_{\rho^j} \chi_{\rho^j}(T_0) \psi_1\|. \quad (2.6)$$

By the monotonicity of χ if $j \geq n$

$$||\Gamma_{\rho^j} \chi_{\rho^j}(T_0) \psi_1|| = ||\chi_{\rho^j}(T_0) \psi_1|| \leq ||\chi_{\rho^n}(T_0) \psi_1|| = ||\Gamma_{\rho^n} \chi_{\rho^n}(T_0) \psi_1||.$$

Thus by (2.5) and (2.6)

$$(a\rho)^2 ||\Gamma_{\rho^n} \chi_{\rho^n}(T_0) \psi_1||^2 \leq \sum_{j=n}^{\infty} (2\delta_0^{-1} a_{j+1})^2 ||\Gamma_{\rho^n} \chi_{\rho^n}(T_0) \psi_1||^2 = d_n ||\Gamma_{\rho^n} \chi_{\rho^n}(T_0) \psi_1||^2$$

where $d_n := (2\delta_0^{-1})^2 \sum_{j=n}^{\infty} |a_{j+1}|^2 \rightarrow 0$ as $n \rightarrow \infty$. Thus for large enough n , $\Gamma_{\rho^n} \chi_{\rho^n}(T_0) \psi_1 = 0$, a contradiction. \square

3 Applications

In the paper [10] a version of the BFS renormalization group approach [1] is developed to prove existence of ground states in certain models. In many models, after one or two initial Feshbach transformations, one arrives at the situation treated in [10], Sections 6–10. From this point on the presentation applies to any model satisfying the hypotheses. We used the method in the spin-boson model [10] and in a model of non-relativistic QED (see [11], [12]) to prove existence of a ground state and analyze some of its properties.

In fact in both of the above models uniqueness is known by methods other than the one presented in this paper. In the spin-boson model one has a representation where e^{-tH} is positivity improving. In our QED model this cannot work in the multi-electron case since we assumed that the electrons were fermions. However, there is a simple method due to Hiroshima and Spohn [13] (or see Section 15.3 of [15]) which gives uniqueness if the Hamiltonian of the model is infrared regular, and in minimally coupled QED, this can be accomplished by a so called Pauli-Fierz gauge transformation. In addition the method of [3] works for minimally coupled QED without any infrared regularization. We would like to point out that there are models where the latter methods will not work but which have unique ground states nevertheless (which can be seen by applying Theorem 2.1 as we do below). In particular if in our QED model Hamiltonian, given in (3.1) below, one drops the $(A_\Lambda(x_j))^2$ terms to obtain a dipole approximation these methods apparently do not give uniqueness but as mentioned in [11] the proof given there works with no essential change for this model to give existence of a ground state. (The main reason that we do not need a Pauli-Fierz transformation to regularize the Hamiltonian in the infra-red is because of the absence of terms of the form $\int w[H_f; k] a(k, \lambda) dk$ and their adjoints in the renormalization group iteration procedure. Of course in models where these terms have good infrared behavior, the method also works (for example see [14])).

We consider the following models and then apply Theorem 2.1. Define the self adjoint

operators

$$\begin{aligned}
H^{(1)} &= \sum_{n=1}^N (p_j + gA_\Lambda(\beta x_j))^2 + V(x_1, \dots, x_N) + H_f \\
H^{(2)} &= \sum_{n=1}^N (p_j^2 + 2gp_j \cdot A_\Lambda(\beta x_j)) + V(x_1, \dots, x_N) + H_f \\
H^{(3)} &= (\sigma_z + 1) \otimes I + I \otimes H_f + g\sigma_x \otimes \phi(f)
\end{aligned} \tag{3.1}$$

The operators $H^{(1)}$ and $H^{(2)}$ act in the Hilbert space $\mathcal{H} = L_a^2(\mathbb{R}^{3N}; \mathcal{F})$ and $\beta, g \in \mathbb{R}$. Here \mathcal{F} is the Fock space for transversally polarized photons and the a indicates anti-symmetry in the N electron coordinates, x_1, \dots, x_N . The potential V is assumed to be invariant under rotations and permutations of the co-ordinates and Kato small with respect to the electron kinetic energy $\sum_{j=1}^N p_j^2$ where $p_j = -i\nabla_{x_j}$. By $A_\Lambda(x)$ we denote the quantized vector potential at position x [15, 11]. The subscript Λ indicates a rotation invariant ultraviolet cut-off. We do not impose an infrared regularization. The operator H_f is the kinetic energy of the photons each of which has dispersion relation $\omega(k) = |k|$. See [11] and [12] for details of the model. It is known that these models have ground states (at least for small g). For $H^{(1)}$ (for all g) see [6] and references given there. Although only $H^{(1)}$ is explicitly treated in [11] the results given there also hold for $H^{(2)}$ by essentially the same but a slightly simpler proof. For the spin boson Hamiltonian, $H^{(3)}$, see [10]. This Hamiltonian acts in $\mathbb{C}^2 \otimes \mathcal{F}$. Here \mathcal{F} is the Fock space over the square integrable functions on \mathbb{R}^3 . The operator $\phi(f) = (a^*(f/\sqrt{\omega})) + (a^*(f/\sqrt{\omega}))^*$ where f is in $L^\infty(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$ and $a^*(\cdot)$ is the usual creation operator. By σ_z and σ_x we denote the Pauli matrices. Let $H_{\text{atom}} = \sum_{j=1}^N p_j^2 + V(x_1, \dots, x_N)$ in $L_a^2(\mathbb{R}^{3N})$. Then we have the following result.

Theorem 3.1. *Suppose $H = H^{(1)}$, $H = H^{(2)}$, or $H = H^{(3)}$ with the definitions given above. Suppose H_{atom} has a non-degenerate ground state in $L_a^2(\mathbb{R}^{3N})$. Then there exists $g_0 > 0$ so that if $g \in [-g_0, g_0]$, H has a non-degenerate ground state.*

Proof. Referring to [11] and [10] after one or two Feshbach transformations the operator $H^{(j)} - E_0^{(j)}$ transforms to an operator unitarily equivalent to an operator $H_1 = T_1 + W_1$ acting in $\mathcal{H}_{\text{red}} := 1_{[0,1]}(H_f)\mathcal{F}$. The null spaces of the operator and its Feshbach transform, H_1 , are isomorphic. Here $E_0^{(j)}$ is the ground state energy of $H^{(j)}$. We have omitted superscripts. T_1 is a real C^1 function of H_f . Thus referring to Theorem 2.1, $\mathcal{H} = \mathcal{F}$ and $T_0 = H_f$. The unitary scale transformation Γ_ρ leaves the vacuum invariant and acts on the usual creation operators as

$$\Gamma_\rho a^*(k) \Gamma_\rho^{-1} = \rho^{-3/2} a^*(\rho^{-1}k)$$

We have omitted polarization indices for $H^{(1)}$ and $H^{(2)}$. The functions χ and $\bar{\chi}$ are assumed to be real and in $C^\infty(\mathbb{R})$ with χ monotone,

$$\chi^2 + \bar{\chi}^2 = 1, \quad \text{supp } \chi \subset (-\infty, 1], \quad \text{and } \chi(x) = 1 \text{ if } x \in [0, 3/4]. \tag{3.2}$$

Thus in Theorem 2.1, $a = 3/4$. In [10], [11], and [12], omitting polarization indices for $H^{(1)}$ and $H^{(2)}$, W_n is given by

$$W_n = \sum_{l+m=1}^{\infty} \int \prod_{j=1}^l a^*(k_j) w_{l,m}^{(n)}[z_n; H_f; k_1, \dots, k_l, \tilde{k}_1, \dots, \tilde{k}_m] \prod_{i=1}^m a(\tilde{k}_i) \frac{dk_1}{|k_1|^{1/2}} \cdots \frac{d\tilde{k}_m}{|\tilde{k}_m|^{1/2}}.$$

The kernels $w_{l,m}^{(n)}[z_n; r; k_1, \dots, k_l, \tilde{k}_1, \dots, \tilde{k}_m]$ are such that $\|W_n\| \leq \epsilon_0/2^n$ (so that W_n satisfies (b) of Theorem 2.1). The quantity z_n is a real spectral parameter with $|z_n| < c_1/2^n$. The operators H_n are defined inductively by (2.3) where ρ is chosen suitably small. T_n is a real C^1 function of H_f satisfying (for $r \in [0, 1]$)

$$|(\partial_r T_n(r) - 1)| \leq \epsilon_0 \quad (3.3)$$

$$|T_n(0) + z_n| \leq \epsilon_0/2^n \quad (3.4)$$

Here ϵ_0 is a small positive number. It follows that

$$T_n(r) = (r + \int_0^r (\partial_s T_n(s) - 1) ds) + (T_n(0) + z_n) - z_n \quad (3.5)$$

so that

$$\|T_n(H_f)\psi\| \geq (1 - \epsilon_0)\|H_f\psi\| - 2\epsilon_0 2^{-n}\|\psi\| \quad (3.6)$$

Thus T_n satisfies hypothesis (a) of Theorem 2.1. The smallness of ϵ_0 and the bounds satisfied by T_n and W_n require $|g|$ to be small and the proof of these bounds requires ρ to be small, in particular $\rho < 3/4$.

In view of the conformal transformation which is part of the renormalization group analysis of [10] we make some remarks to justify (2.3). In [10] operator functions of a complex variable, $\tilde{H}_n(z) = \tilde{T}_n(z) + \tilde{W}_n(z)$ are a crucial part of the analysis and the Hamiltonians H_n are given by evaluating $\tilde{H}_n(z)$ at the real spectral parameter z_n , $H_n = \tilde{H}_n(z_n)$ where z_n is given by a certain limit. We have

$$z_n = J_n^{-1}(z_{n+1}) \quad (3.7)$$

where J_n is a conformal transformation (see [10] for further definitions and details). We have

$$\tilde{H}_{n+1}(J_n(z)) = \rho^{-1} \Gamma_\rho \tilde{F}_n(z) \Gamma_\rho^{-1}$$

where $\tilde{F}_n(z)$ is given by (2.1) and (2.2) with H_n replaced by $\tilde{H}_n(z)$ and similarly for T_n and W_n . Substituting $z = z_n$ and noting the composition law (3.7), we see that H_{n+1} and H_n are indeed related by (2.3).

□

A Appendix

In this appendix we show that Theorem 2.1 can be used to obtain uniqueness in the framework of operator theoretic renormalization as defined in [1]. To this end we recall the setup of [1]. Let $I = [0, 1]$, let B_1 denote the unit ball in \mathbb{R}^d , and let $D_{1/2} := \{z \in \mathbb{C} \mid |z| \leq 1/2\}$. Here the Fock space \mathcal{F} is over the space of square integrable functions on \mathbb{R}^d . Let $\mathcal{W}_{m,n}^\#$ denote the Banach space of functions $B_1^m \times B_1^n \rightarrow C^1(I)$, which are a.e. defined, totally symmetric under the interchange of components in B_1^m respectively B_1^n , and which satisfy the norm bound

$$\|w_{m,n}\|_\mu^\# := \|w_{m,n}\|_\mu + \|\partial_r w_{m,n}\|_\mu < \infty,$$

where

$$\|w_{m,n}\|_\mu := \left[\int_{B_1^{m+n}} \sup_{r \in I} \left| w_{m,n}[r; k_1, \dots, k_m, \tilde{k}_1, \dots, \tilde{k}_n] \right|^2 \prod_{i=1}^m \frac{d^d k_i}{|k_i|^{3+2\mu}} \prod_{j=1}^n \frac{d^d \tilde{k}_j}{|\tilde{k}_j|^{3+2\mu}} \right]^{1/2}$$

for some $\mu > 0$. For $0 < \xi < 1$ one defines, $\mathcal{W}_\xi^\# := \bigoplus_{m+n \geq 0} \mathcal{W}_{m,n}^\#$, to be the Banach space of sequences $\underline{w} = (w_{m,n})_{m+n \geq 1}$ obeying

$$\|\underline{w}\|_{\mu,\xi}^\# := \sum_{m+n \geq 1} \xi^{-(m+n)} \|w_{m,n}\|_\mu^\# < \infty.$$

We define

$$\begin{aligned} W_{m,n}[w_{m,n}] &:= 1_I(H_f) \int_{B_1^{m+n}} \prod_{j=1}^m a^*(k_j) w_{m,n}[H_f; k_1, \dots, k_m, \tilde{k}_1, \dots, \tilde{k}_n] \prod_{l=1}^n a(\tilde{k}_l) \\ &\quad \times \prod_{j=1}^m \frac{d^d k_j}{|k_j|^{1/2}} \prod_{l=1}^n \frac{d^d \tilde{k}_l}{|\tilde{k}_l|^{1/2}} 1_I(H_f), \end{aligned}$$

where $a^*(k)$ and $a(k)$ denote the usual creation and annihilation operator. In [1] it is shown that $H(\underline{w}) = \sum_{m+n \geq 0} W_{m,n}[w_{m,n}]$ defines a bounded operator on $\mathcal{H}_{\text{red}} := 1_I(H_f)\mathcal{F}$, with bound

$$\|H(\underline{w})\| \leq \|\underline{w}\|_{\mu,\xi}^\#. \quad (\text{A.1})$$

We define the polydics $\mathcal{D}(\epsilon, \delta)$ to consist of the analytic¹ functions $\underline{w}[\cdot] : D_{1/2} \rightarrow \mathcal{W}_\xi^\#$, with

$$\begin{aligned} \sup_{z \in D_{1/2}} \sup_{r \in I} |\partial_r w_{0,0}[z; r] - 1| &\leq \epsilon, \\ \sup_{z \in D_{1/2}} |w_{0,0}[z; 0] + z| &\leq \delta, \quad \sup_{z \in D_{1/2}} \|(w_{m,n}[z])_{m+n \geq 1}\|_{\mu,\xi}^\# \leq \delta. \end{aligned}$$

¹Analytic on the closed set $D_{1/2}$ means that the function is analytic in an open neighborhood of $D_{1/2}$.

Let $0 < \rho < 1/2$, and assume $\underline{w} \in \mathcal{D}(\rho/8, \rho/8)$. In [1] it is shown that the following map is biholomorphic

$$E_\rho[\cdot] : \mathcal{U}[w] \rightarrow D_{1/2}, \quad z \mapsto -\rho^{-1}w_{0,0}[z; 0],$$

where $\mathcal{U}[w] := \{z \in D_{1/2} \mid |w_{0,0}[z; 0]| \leq \rho/2\}$. Assume $\chi, \overline{\chi} \in C^\infty(\mathbb{R}; [0, 1])$ satisfy (3.2) and that χ is monotone. Define $\chi_\rho := \chi(\rho^{-1}H_f)$ and $\overline{\chi}_\rho := \overline{\chi}(\rho^{-1}H_f)$. Let $\zeta \in D_{1/2}$, and set

$$T(r) := w_{0,0}[E_\rho^{-1}(\zeta); r], \quad W := \sum_{m+n \geq 1} W_{m,n} [w_{m,n}[E_\rho^{-1}(\zeta)]] .$$

Then, as shown in [1], the map $H_{\overline{\chi}_\rho} := T(H_f) + \overline{\chi}_\rho W \overline{\chi}_\rho$ is bounded invertible on the range of $\overline{\chi}_\rho$ and there exists a unique so called renormalized kernel, $\mathcal{R}_\rho(\underline{w})$, such that

$$H(\mathcal{R}_\rho(w)[\zeta]) = \frac{1}{\rho} \Gamma_\rho \left(T(H_f) + \chi_\rho W \chi_\rho - \chi_\rho W \overline{\chi}_\rho H_{\overline{\chi}_\rho}^{-1} \overline{\chi}_\rho W \chi_\rho \right) \Gamma_\rho^*. \quad (\text{A.2})$$

Furthermore, for fixed $\mu > 0$, there exist $\rho, \xi, \epsilon_0 > 0$ such that

$$\mathcal{R}_\rho : \mathcal{D}(\epsilon, \delta) \rightarrow \mathcal{D}(\epsilon + \delta/2, \delta/2) \quad (\text{A.3})$$

for all $\epsilon, \delta \in [0, \epsilon_0]$. The existence part of the following theorem has been proven in [1, Theorem 3.12], i.e., with assertion l.h.s. (A.4) \geq r.h.s. (A.4). Using Theorem 2.1 one can show that one has uniqueness, i.e., l.h.s. (A.4) \leq r.h.s. (A.4).

Theorem A.1. *Fix $\mu > 0$, and choose $\rho, \xi, \epsilon_0 > 0$ sufficiently small such that (A.3) holds for all $0 \leq \epsilon \leq \epsilon_0$ and $0 \leq \delta \leq \epsilon_0$. Suppose that $\underline{w} \in \mathcal{D}(\epsilon_0/2, \epsilon_0/2)$. Then the complex number $e_{(0,\infty)} \in D_{1/2}$ defined in (A.6) is an eigenvalue of $H(\underline{w})$, in the sense that*

$$\dim \ker \{H(\underline{w}[e_{(0,\infty)}])\} = 1. \quad (\text{A.4})$$

Proof. In view of [1, Theorem 3.12] it remains to show $\dim \ker \{H(\underline{w}[e_{(0,\infty)}])\} \leq 1$. To this end we combine the proof given in [1] with Theorem 2.1. The contraction property (A.3) allows to iterate the renormalization transformation, and hence for $n \in \mathbb{N}_0$ the kernels $\underline{w}^{(n)} := \mathcal{R}_\rho^n(\underline{w})$ satisfy

$$\underline{w}^{(n)} \in \mathcal{D}(\epsilon_0, 2^{-n}\epsilon_0). \quad (\text{A.5})$$

We define $E_n[z] := w_{0,0}^{(n)}[z; 0]$, $J_n[z] = \rho^{-1}E_n[z]$, and for $n \leq m \in \mathbb{N}_0$

$$e_{(n,m)} := J_{(n)}^{-1} \circ \cdots \circ J_{(m)}^{-1}[0].$$

Using property (A.5) it was shown in [1] (c.f. Eq. (3.146)) that the following limit exists

$$z_n := e_{(n,\infty)} := \lim_{m \rightarrow \infty} e_{(n,m)}. \quad (\text{A.6})$$

It follows from the definition of the renormalization transformation (A.2) that

$$H_n = \frac{1}{\rho} \Gamma_\rho F_{n-1} \Gamma_\rho^*,$$

where

$$\begin{aligned}
H_n &:= H(\underline{w}^{(n)}[e_{(n,\infty)}]), \\
T_n(r) &:= w_{0,0}^{(n)}[e_{(n,\infty)}; r], \quad W_n := \sum_{m+l \geq 1} W_{m,l} [w_{m,l}^{(n)}[e_{(n,\infty)}]], \\
F_n &:= T_n(H_f) + \chi_\rho W_n \chi_\rho - \chi_\rho W_n \overline{\chi}_\rho (T_n(H_f) + \overline{\chi}_\rho W_n \overline{\chi}_\rho)^{-1} \overline{\chi}_\rho W_n \chi_\rho.
\end{aligned}$$

It follows from (A.5), that (3.3) and (3.4) hold. Then using the decomposition (3.5) we find (3.6). Thus $T_n(H_f)$ satisfies Hypothesis (a) of Theorem 2.1 with $T_0 = H_f$. Now Hypthesis (b) of Theorem 2.1 follows from

$$\|W_n\| \leq \epsilon_0 2^{-n}. \quad (\text{A.7})$$

Ineq. (A.7) can be shown using (A.1) and (A.5). The theorem now follows as a consequence of Theorem 2.1. \square

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